GALOIS THEORY TOPIC II
COMPLEX NUMBERS

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1. Complex Algebra

The set of *complex numbers* is

$$\mathbb{C} = \{a + ib \mid b \in \mathbb{R}, i^2 = -1\}.$$

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = x_1 + iy_1$ and $z_2 = x_2 iy_2$ for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Define addition and multiplication in $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2);$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2).$$

Thus to add or multiply complex numbers, treat i like a variable, add or multiply, replace i^2 with -1, and combine like terms.

One can show that these operations have the following properties:

(F1) $a + b = b + a$ for every $a, b \in \mathbb{C}$;
(F2) $(a + b) + c = a + (b + c)$ for every $a, b, c \in \mathbb{C}$;
(F3) there exists $0 \in \mathbb{C}$ such that $a + 0 = a$ for every $a \in \mathbb{C}$;
(F4) for every $a \in \mathbb{C}$ there exists $b \in \mathbb{C}$ such that $a + b = 0$;
(F5) $ab = ba$ for every $a, b \in \mathbb{C}$;
(F6) $(ab)c = a(bc)$ for every $a, b, c \in \mathbb{C}$;
(F7) there exists $1 \in \mathbb{C}$ such that $a \cdot 1 = a$ for every $a \in \mathbb{C}$;
(F8) for every $a \in \mathbb{C} \setminus \{0\}$ there exists $c \in \mathbb{C}$ such that $ac = 1$;
(F9) $a(b + c) = ab + ac$ for every $a, b, c \in \mathbb{C}$.

Together, these properties state that $\mathbb{C}$ is a *field*. Note that

- $0 = 0 + i0$;
- $1 = 1 + i0$;
- $-(x + iy) = -x + i(-y) = -x - iy$;
- $(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}$.

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2. Complex Geometry

Let $z = x + iy$ be an arbitrary complex number. The *real part* of z is $\Re(z) = x$. The *imaginary part* of z is $\Im(z) = y$. We view $\mathbb{R}$ as the subset of $\mathbb{C}$ consisting of those elements whose imaginary part is zero.

We graph complex number on the xy-plane, using the real part as the first coordinate and the imaginary part as the second coordinate. Under this interpretation, the set $\mathbb{C}$ becomes a real vector space of dimension two, with scalar multiplication given by complex multiplication by a real number. We call this vector space the *complex plane*.

Thus the geometric interpretation of complex addition is vector addition.

Let $z = x + iy$ be an arbitrary complex number. The *conjugate* of z is

$$\overline{z} = x - iy.$$

This is the mirror image of z under reflection across the real axis. Note that

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\Re(z).$$

The *modulus* of z is

$$|z| = \sqrt{x^2 + y^2}.$$

This is the length of z as a vector. Note that

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

The *angle* of z, denoted by $\angle(z)$, is the angle between the vectors $(1, 0)$ and (x, y) in the real plane $\mathbb{R}^2$; this is well-defined up to a multiple of 2π.

Let $r = |z|$ and $\theta = \angle(z)$. Then $x = r\cos\theta$ and $y = r\sin\theta$. Define a function

$$\operatorname{cis} : \mathbb{R} \to \mathbb{C} \quad \text{by} \quad \operatorname{cis}(\theta) = \cos\theta + i\sin\theta.$$

Then $z = r\operatorname{cis}(\theta)$; this is the *polar representation* of z.

Recall the trigonometric formulae for the cosine and sine of the sum of angles:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad \text{and} \quad \sin(A+B) = \cos A\sin B + \sin A\cos B.$$

Let $z_1 = r_1\operatorname{cis}(\theta_1)$ and $z_2 = r_2\operatorname{cis}(\theta_2)$. Then

$$\begin{aligned}
z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\
&= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))\\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))\\
&= r_1 r_2\operatorname{cis}(\theta_1 + \theta_2).
\end{aligned}$$

Thus the geometric interpretation of complex multiplication is:

(a) The radius of the product is the product of the radii;
(b) The angle of the product is the sum of the angles.

Example 1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 2z$. Then f dilates the complex plane by a factor of 2.

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz$. Then f rotates the complex plane by 90 degrees.

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = (1 + i)z$. Note that $|1 + i| = \sqrt{2}$ and $\angle(1 + i) = \frac{\pi}{4}$. Then f dilates the complex plane by a factors of $\sqrt{2}$ and rotates it by 45 degrees.

3. Complex Powers and Roots

A special case of complex multiplication is exponentiation by a natural number; a simple proof by induction shows that

Theorem 1. (DeMoivre's Theorem)
Let $\theta \in \mathbb{R}$. Then

$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

Let $z = r\operatorname{cis}(\theta)$ and let $n \in \mathbb{N}$. Then $z^n = r^n \operatorname{cis}(n\theta)$.

The *unit circle* in the complex plane is

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Note that if $u_1, u_2 \in \mathbb{U}$, then $u_1u_2 \in \mathbb{U}$.

Let $\zeta \in \mathbb{C}$ and suppose that $\zeta^n = 1$. We call ζ an n^{th} *root of unity*. If $\zeta^m \neq 1$ for $m \in \{1, \ldots, n-1\}$, we call ζ a *primitive* n^{th} *root of unity.*

Let $\zeta = \operatorname{cis}(\frac{2\pi}{n})$. Then $\zeta^n = \operatorname{cis}(n\frac{2\pi}{n}) = \operatorname{cis}(2\pi) = 1$; one sees that ζ is a primitive n^{th} root of unity. Thus primitive roots of unity exist for every n. As m ranges from 0 to $n-1$, we obtain distinct complex numbers ζ^m, all of which are n^{th} roots of unity. These are all of the n^{th} roots of unity; thus for each $n \in \mathbb{N}$, there are *exactly* n distinct n^{th} roots of unity.

If one graphs the n^{th} roots of unity in the complex plane, the points lie on the unit circle and they are the vertices of a regular n-gon, with one vertex always at the point $1 = 1 + i0$.

Let $z = r\operatorname{cis}(\theta)$. Then z has exactly n distinct n^{th} roots; they are

$$\sqrt[n]{z} = \zeta^m \sqrt[n]{r}\operatorname{cis}\left(\frac{\theta}{n}\right), \quad \text{where} \quad \zeta = \operatorname{cis}\left(\frac{2\pi}{n}\right) \text{ and } m \in \{0, \ldots, n-1\}.$$

The algebraic importance of the complex numbers, and the original motivation for their study, is exemplified by the next theorem. This was first conjectured in the 1500's, but was not proven until the doctoral dissertation of Carl Friedrich Gauss in 1799 at the age of 22. Incidentally, was the first to prove the constructibility of a regular 17-gon, at an even earlier age.

Theorem 2. (The Fundamental Theorem of Algebra)
Every polynomial with complex coefficients has a zero in $\mathbb{C}$.

From this, it follows that every polynomial with complex coefficients factors completely into the product of linear polynomials with complex coefficients.

4. Complex Analysis

The *distance* between complex number z_1 and z_2 is $|z_1 - z_2|$. This is standard distance in the complex plane, and allows us to precisely define what it means for a complex function to be continuous or differentiable. We discuss this briefly.

Let $f : \mathbb{C} \to \mathbb{C}$. We say that f is *continuous* at z_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

Let $f : \mathbb{C} \to \mathbb{C}$ and $L \in \mathbb{C}$. We say that the *limit* of f as z approaches z_0 is L, and write $\lim_{z \to z_0} f(z) = L$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$.

We say that f is *differentiable* at z_0 if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Complex differentiability has some amazing consequences; for example, it can be shown that every complex differentiable function is analytic, which means that it can be written as a power series.

Define the complex exponential function

$$\exp : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This is motivated by the Taylor expansion of the real exponential function.

Define the complex sine function by

$$\sin : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Define the complex cosine function by

$$\cos : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

Note that exp, sin, and cos, when restricted to $\mathbb{R} \subset \mathbb{C}$, are defined so as to be consistent with other definitions of these real functions.

Define $\log : \mathbb{C} \to \mathbb{C}$ to be an inverse function of exp. Let $w, z \in \mathbb{C}$. We define w^z by

$$w^z = \exp(z \log(w)).$$

Thus $\exp(z) = e^z$.

One computes that

$$\exp(iz) = \cos(z) + i \sin(z).$$

In particular, if z is the complex number $i\theta$, where $\theta \in \mathbb{R}$, we have

Theorem 3. (Euler's Theorem) *Let $\theta \in \mathbb{R}$. Then*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

5. Construction of Complex Numbers

Let $z \in \mathbb{C}$. We say that z is *constructible* if the the real and imaginary parts of z are constructible real numbers.

This is equivalent to the condition that the point z, viewed as a point in the complex plane, can be constructed from the points corresponding to the complex numbers 0 and 1.

To see this, suppose that z is constructible; this means that z can be constructed from the complex numbers 0 and 1; we construct the real and imaginary parts of z from the set $\{0, 1, z\}$. The line through zero and one is the real axis. We can construct a line through zero and perpendicular to the real axis; this line is the imaginary axis. Thus we can construct a lines through z which are perpendicular to the real and imaginary axis, so we can construct the points of intersection of these lines with the real and imaginary axis. The length of the line segments so produced are the real and imaginary parts of z.

On the other hand, suppose that the real and imaginary parts of z are constructible real numbers. By transference of distance, we can construct points on the real and imaginary axis whose distance from zero is the real and imaginary parts of z, then take perpendicular lines through these points. The intersection of these lines will be z, so z is constructible.

For example, if $z = a + 0i$, then we plot z on the Cartesian plane at the point $(a, 0)$; we know that this point is constructible if and only if the real number a is a constructible real numbers. Thus our definition matches our previous definition for the case of real numbers.

6. Exercises

The *rectangular form* of a complex number is $z = a + bi$.
The *polar form* of a complex number is $z = r \operatorname{cis} \theta$.

Exercise 1. Let $z = 7 - 2i$ and $w = 5 + 3i$.
Compute the following, expressed in rectangular form.

(a) $z + w$
(b) $3z - 8w$
(c) zw
(d) $\frac{z}{w}$
(e) $\overline{z}$ and $|z|$

Exercise 2. Find the rectangular and polar forms of all sixth roots of unity.

Exercise 3. Find the rectangular and polar forms of all solutions to the equation $z^6 - 8 = 0$.

Exercise 4. Find the rectangular and polar forms of all solutions to the equation $z^6 - a = 0$, where $a = \sqrt{3} + i$.

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